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6 LOCAL ASYMPTOTIC NORMALITY FOR PROGRESSIVELY CENSORED LIKELIHOOD RATIO STATISTICS AND APPLICATIONS*

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Summary. Let $X_{n,1} \leq X_{n,2} \leq \ldots \leq X_{n,n}$ be the ordered variables corresponding to a random sample of size n with respect to a family of probability measures $\{P_\theta : \theta \in \Theta\}$ where Θ is an open subset of the real line. In many practical situations the $X_{n,i}$ are the observables and experimentation must be curtailed prior to $X_{n,n}$. If τ_n is a stopping variable adapted to the σ-fields $\{\sigma(X_{n,1},\ldots,X_{n,k}) : 1 \leq k \leq n\}$ and $P_{n,\theta}$ the projection of P_θ onto $\sigma(X_{n,1},\ldots,X_{n,\tau_n})$, the local asymptotic normality of the stopped progressively censored likelihood ratio statistics $\Lambda_{n,\tau_n} = dP_{n,\theta_n}/dP_{n,\theta}$ is established with θ, $\theta_n = \theta + un^{-\frac{1}{2}} \in \Theta$ and θ,u held fixed, under certain conditions on the underlying distribution and on τ_n. Conditions are also given to ensure the local asymptotic normality of likelihood ratio statistics where the underlying observations are given in a series scheme.

14 RM-394

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1. Introduction.

There have been in recent times several investigations were made of the notion of local asymptotic normality (LAN) of a family of probability distributions. The usefulness of this concept in problems of the theory of asymptotic estimation and hypothesis testing ~~has been~~ was demonstrated, ~~in the papers of~~ LeCam and in LeCam (1960) can be found a comprehensive examination of sets of conditions can be found ensuring the LAN for different families of probability distributions. Much attention has been focused on distributions connected with sequences of independent and identically distributed (iid) observations or those associated with a homogeneous Markov chain. See, for example Hájek (1972), Roussas (1972), Ibragimov and Khas'minskii (1972), Inagaki and Ogata (1975). The case of independent but non-identically distributed observations is also discussed in Phillipou and Roussas (1973) and Ibragimov and Khas'minskii (1975).

In this paper ~~we shall present~~ conditions are presented ensuring the LAN for distributions connected with certain observations, which are neither independent nor identically distributed, when a class of stopping variables is incorporated in a natural way into the basic framework. The primary motivation for our study is the following situation which arises naturally in the context of clinical trials and life tests. The typical feature encountered here is that even though the survival times of $n \geq 1$ specimens under a life test may be iid random variables (rv) $X_1, \ldots, X_n$, with respect to a family of probability measures $\{P_\theta: \theta \in \Theta\}$ the actual observables are the dependent ordered variables $X_{n,1}, \ldots, X_{n,n}$ of the responses of the sample. Limitations on time and cost and ethical reasons invariably force termination of experimentation before the last observable $X_{n,n}$ is recorded and thus, in practice, sampling may be terminated after a pre-specified duration of time (truncation)

or once a pre-specified proportion of units have responded (censoring). Alternatively, in progressive censoring, the experiment is monitored from the onset with the data continually updated so that at any stage if the current accumulated evidence suggests a clear statistical decision, then experimentation can be curtailed with a concomitant reduction in cost and sacrifice of lives of experimental units. Thus we are led naturally to the incorporation of a class of stopping variables τ_n into our basic framework, where for each $n \geq 1$, τ_n is defined in terms of the recorded observables among $X_{n,1},\ldots,X_{n,n}$. If $P_{n,\theta}$ denotes the restriction of P_θ to the σ-field generated by $X_{n,1},\ldots,X_{n,\tau_n}$, the asymptotic behavior of the likelihood ratios $dP_{n,\theta_n}/dP_{n,\theta}$ for certain sequences of parameter values θ_n will be of interest. It is to this question that this paper is devoted. Note that when τ_n is degenerate at n, for all n, $dP_{n,\theta_n}/dP_{n,\theta}$ reduce to the likelihood ratio for the n iidrv $X_1,\ldots,X_n$ and the limiting distribution of these ratios has been studied with $\theta_n = \theta + un^{-\frac{1}{2}}$. An investigation of the general situation was initiated by Sen (1976) for certain likelihood ratio processes. Here τ_n is assumed degenerate at r_n and r_n/n tends to $\alpha \in (0,1]$ as $n \to \infty$. By considering random τ_n we allow for wider applicability and the results that we obtain are stronger and assume fewer regularity conditions.

The basic notation and assumptions are summarized in Section 2. In Section 3, the main theorems in this paper are stated and their proofs taken up in Section 4 after some auxiliary lemmata are established. In the last section we make a few comments on our assumptions and discuss an extension of our results to likelihood ratio statistics when the observations follow a series scheme.

2. Preliminary notions and the main theorems.

Let $\{X_i : i \geq 1\}$ be a sequence of iidrv's whose probability distribution ν_θ on the Borel line $(R,\mathcal{B})$ depends on a parameter θ belonging to an open subset Θ of R. We suppose that the family of measures $\{\nu_\theta : \theta \in \Theta\}$ is dominated by Lebesgue measure μ on $(R,\mathcal{B})$ and write $f_\theta(\cdot) = d\nu_\theta/d\mu$ for a version of the probability density function (pdf) and $F_\theta(\cdot)$ for the corresponding distribution function (df). Let $(R_j,\mathcal{B}_j)$, $j \geq 1$ be copies of the Borel line and set $(X^*,A^*) = \Pi_{j=1}^{\infty} (R_j,\mathcal{B}_j)$ with P_θ denoting the product measure of the ν_θ induced on A^*. E_θ will denote the expectation evaluated with respect to P_θ.

We envisage a clinical trial or life test experiment in which the X_i denote survival or response times and consequently they are nonnegative and the observable variables for a sample of $n \geq 1$ specimens are the order statistics $X_{n,1} <\ldots< X_{n,n}$ corresponding to $X_1,\ldots,X_n$. By the continuity of F_θ ties among the observables may be disregarded with probability one.

For simplicity in script we denote by

$$Z_k = X_{n,k},\ \underset{\sim}{Z}^{(k)} = (Z_1,\ldots,Z_k),\ 1 \leq k \leq n;\ Z_0 = \underset{\sim}{Z}^{(0)} = 0. \tag{2.1}$$

The σ-field generated by $\underset{\sim}{Z}^{(k)}$ is written $\mathcal{B}_{n,k}$ and $\mathcal{B}_{n,0}$ is the trivial σ-field. For each $n \geq 1$, let τ_n be a stopping variable adapted to $\{\mathcal{B}_{n,k} : 1 \leq k \leq n\}$. We denote by $P_{n,\theta}$ the projection of P_θ on $\mathcal{B}_{n,\tau_n} = \sigma(\underset{\sim}{Z}^{(\tau_n)})$. The family of probability measures $\{P_{n,\theta} : \theta \in \Theta\}$ is said to be locally asymptotically normal (LAN) at $\theta_0 \in \Theta$ if for some positive nonstochastic sequence $\{\varphi_n : n \geq 1\}$ we have for each $u \in R$

$$\frac{dP_{n,\theta_0+u\varphi_n^{-1}}}{dP_{n,\theta_0}} = \exp\{u\, W_n(\theta_0) - \frac{1}{2}\, u^2 + \delta_n(u,\theta_0)\} \tag{2.2}$$

where $\{W_n(\theta_0) : n \geq 1\}$ converges weakly under P_{θ_0} to a Gaussian (0,1) variable and, for each u, $\{\delta_n(u,\theta_0) : n \geq 1\}$ converges in P_{θ_0}-probability to zero.

Now for each k, $1 \leq k \leq n$ let $P^{(k)}_{n,\theta}$ denote the projection of P_θ onto $\mathcal{B}_{n,k}$. Then with respect to Lebesgue measure μ_k in R^k the joint pdf of $\underset{\sim}{Z}^{(k)}$ is given by

$$p_\theta(\underset{\sim}{z}^{(k)},n) = \{n!/(n-k)!\}\{\prod_{i=1}^{k} f_\theta(z_i)\}\{1-F_\theta(z_k)\}^{n-k} \tag{2.3}$$

defined on $A_{n,k} = \{\underset{\sim}{z}^{(k)} : 0 < z_1 <\ldots< z_k < \infty\}$, and the conditional pdf of Z_k given $\mathcal{B}_{n,k-1}$ is

$$q_\theta(z_k|\mathcal{B}_{n,k-1}) = (n-k+1)f_\theta(z_k)\{1-F_\theta(z_k)\}^{n-k}/\{1-F_\theta(Z_{k-1})\}^{n-k+1} \tag{2.4}$$

defined for $z_k > Z_{k-1}$. Let θ_0 be a fixed but otherwise arbitrary element of Θ and consider the sequence $\{\theta_n\}$ where

$$\theta_n = \theta_0 + un^{-\frac{1}{2}},\ u \in R. \tag{2.5}$$

For $\theta_n \in \Theta$ and for each k, $1 \leq k \leq n$, we define the progressively censored likelihood ratio statistics (PCLRS) by

$$\frac{dP^{(k)}_{n,\theta_n}}{dP^{(k)}_{n,\theta_0}} = \Lambda_{n,k}(u) = p_{\theta_n}(\underset{\sim}{Z}^{(k)},n)/p_{\theta_0}(\underset{\sim}{Z}^{(k)},n). \tag{2.6}$$

Then $dP_{n,\theta_n}/dP_{n,\theta_0} = \Lambda_{n,\tau_n}$ and we are interested in the asymptotic behavior of Λ_{n,τ_n}. In order to state our assumptions we need the hazard rate

$$r_\theta(x) = f_\theta(x)/\{1-F_\theta(x)\} \tag{2.7}$$

and survival function

$$G_\theta(x) = 1-F_\theta(x). \tag{2.8}$$

For any $\mathcal{B}$-measurable nonnegative function $h_\theta(x)$, let $\dot{h}_\theta(x) = \frac{\partial}{\partial\theta}(\log h_\theta(x))$. We now state the assumptions under which we derive the LAN of $\{P_{n,\theta} : \theta \in \Theta\}$ at θ_0.

(A1) For all θ in some neighborhood N_{θ_0} of $\theta_0 \in \Theta$, $f_\theta(x) > 0$ for all $x \in R^+ = [0,\infty)$ and there exists a μ-integrable function U such that for μ-almost all $x \in R^+$, $\theta \to f_\theta(x)$ is continuously differentiable and $|\frac{\partial f_\theta}{\partial\theta}| \leq U$, for all $\theta \in N_{\theta_0}$ and almost all $x \in R^+$.

(A2) For μ-almost all $x \in R^+$, $x \to \dot{r}_{\theta_0}(x)$ is differentiable.

(A3) There exists a number $\delta > 0$ such that $E_{\theta_0}|\dot{r}_{\theta_0}(X)|^{2+\delta} < \infty$ and $E_{\theta_0}|\dot{f}_{\theta_0}(X)|^{2+\delta} < \infty$.

(A4) There exists a constant $\alpha \in (0,1]$ such that $n^{-1}\tau_n \to \alpha$ in P_{θ_0}-probability. Suppose $J_\alpha = J_\alpha(\theta_0) = \int_0^{F_{\theta_0}^{-1}(\alpha)} \dot{r}_{\theta_0}^2(x)\,dF_{\theta_0}(x) > 0$.

(A5) For each k, $1 \leq k \leq n$, $\theta \to \int_{z_{k-1}}^{\infty} q\,(z|\mathcal{B}_{n,k-1})d\mu(z)$ is differentiable through the integral sign at θ_0.

(A6) For each $u \in R$

$$\lim_{n\to\infty} E_{\theta_0}\{\sup_{|\theta-\theta_0|\leq|u|n^{-1/2}} n^{-1}\sum_{i=1}^{n}\int_{Z_{i-1}}^{\infty} d\mu[\frac{\partial}{\partial\theta}\{q_\theta(z|\mathcal{B}_{n,i-1})\}^{1/2} - \frac{\partial}{\partial\theta_0}\{q_{\theta_0}(z|\mathcal{B}_{n,i-1})\}^{1/2}]^2\} = 0,$$

where $\frac{\partial}{\partial\theta_0}\{q_{\theta_0}(z|\mathcal{B}_{n,i-1})\}^{1/2} = (\frac{\partial}{\partial\theta}\{q_\theta(z|\mathcal{B}_{n,i-1})\}^{1/2})_{\theta=\theta_0}$.

Assumption (A2) provides for a Lyapunov Condition for the derivatives $\{\dot{q}_\theta(Z_k|\mathcal{B}_{n,k-1}),\ 1 \leq k \leq n\}$ which form a martingale in view of (A5). Assumption (A6), given in terms of the square root of the conditional pdf is a weak continuity condition. Our restriction on the growth of τ_n is

(A4). J_α is the analog of the Fisherian Information for our framework. Indeed for $\alpha = 1$ this reduces to $J_1 = E_{\theta_0}(\dot{f}_{\theta_0}(X))^2$ which is the Fisher Information of an unit sample from the distribution F_{θ_0}.

For each k, $1 \leq k \leq n$ denote by

$$\xi_{n,k} \equiv \xi_{n,k}(\underset{\sim}{Z}^{(k)}, \theta_0) = \dot{p}_{\theta_0}(Z^{(k)}, n) \tag{2.9}$$

and let

$$J_{n,k} = E_{\theta_0}\{\xi^2_{n,k}\}. \tag{2.10}$$

With a slight abuse of standard notation we set

$$J_{n,\tau_n} = E_{\theta_0}\{\xi^2_{n,\tau_n}\}. \tag{2.11}$$

Now if

$$\xi^*_{n,k} \equiv \xi^*_{n,k}(\underset{\sim}{Z}^{(k)}, \theta_0) = \dot{q}_{\theta_0}(Z_k | \mathcal{B}_{n,k-1}),\ 1 \leq k \leq n \tag{2.12}$$

and $\sigma^{*2}_{n,k} = E_{\theta_0}(\xi^{*2}_{n,k} | \mathcal{B}_{n,k-1})$ let

$$V_{n,k} = \sum_{i=1}^{k} \sigma^{*2}_{n,i}\ . \tag{2.13}$$

Define the sequence of statistics $\{W_{n,\tau_n} : n \geq 1\}$ by

$$W_{n,\tau_n} = \xi_{n,\tau_n} / J^{\frac{1}{2}}_{n,\tau_n}\ . \tag{2.14}$$

3. Main Theorems.

Define processes $u \to \Lambda^{\pm}(u)$, $u \in R$ by

$$\Lambda^{\pm}(u) = \exp\{u\, J^{\frac{1}{2}}_{\alpha}\zeta \pm \frac{1}{2} u^2 J_\alpha\} \tag{3.1}$$

where ζ is a standard normal variable. Then we have

Theorem 3.1. *Under assumptions A1-A6 for each $u \in R$ the following representation holds.*

$$\Lambda_{n,\tau_n}(u) = \exp\{u\, J_\alpha^{\frac{1}{2}}\, W_{n,\tau_n} - \frac{1}{2}\, u^2 J_\alpha + \delta_n\} \tag{3.2}$$

where $L(W_{n,\tau_n} | P_{\theta_0}) \to N(0,1)$ *and for each* $u \in R$, $\delta_n(u) \to 0$ *in* P_{θ_0}-*probability.*

Therefore we may say the family of probability measures $\{P_{n,\theta} : \theta \in \Theta\}$ is LAN at $\theta_0 \in \Theta$. For statistical applications it is necessary to investigate the behavior of Λ_{n,τ_n} under both P_θ and P_{θ_n}.

Theorem 3.2. *Under assumptions* A1-A6 *for each* $u \in R$

$$L(\Lambda_{n,\tau_n}(u) | P_{\theta_0}) \to L(\Lambda^-(u)) \tag{3.3}$$

and

$$L(\Lambda_{n,\tau_n}(u) | P_{\theta_n}) \to L(\Lambda^+(u)). \tag{3.4}$$

In the framework which we have described the observable variables are the ordered observations $Z_1,\ldots,Z_n$ corresponding to the sample $X_1,\ldots,X_n$ and therefore it is appropriate to formulate statistical procedures in terms of the Z_i rather than the X_i themselves. Accordingly we consider a sequence of statistics $\{T_{n,k} = T_{n,k}(\underset{\sim}{Z}^{(k)}), 1 \le k \le n\}$ and the stopped sequence $\{T_{n,\tau_n} : n \ge 1\}$ which then leads to the adoption of the sequential plan (T_{n,τ_n}, τ_n). We shall not pursue here the interesting question of the optimality (in some sense) of such procedures but, as an application of our previous theorems we establish a result which may be connected with its resolution.

Let μ_k denote Lebesgue measure on (R^k, B^k) and $E_{n,k} \in B^k$ be the set on which $\tau_n = k$. Theorem 3.3 is the Cramér-Rao inequality and Theorem 3.4 gives a lower bound for the asymptotic variance of certain T_{n,τ_n}.

Theorem 3.3. *Suppose conditions* A1-A5 *hold for each* $\theta_0 \in \Theta$. *For each* k, $1 \le k \le n$ *suppose* $E_\theta T^2_{n,k} < \infty$ *and* $\int_{E_{n,k}} T_{n,k}(\underset{\sim}{z}^{(k)})p_\theta(\underset{\sim}{z}^{(k)},n)d\mu_k(\underset{\sim}{z}^{(k)})$ *be differentiable through the integral sign with respect to* θ. *Then for each*

$\theta \in \Theta$

$$\mathrm{Var}_\theta(T_{n,\tau_n}) \geq \left(\frac{d}{d\theta}\lambda_n(\theta)\right)^2 / J_{n,\tau_n}(\theta) \tag{3.5}$$

where $\lambda_n(\theta) = E_\theta T_{n,\tau_n}$. *If equality obtains for sufficiently large* n *and* $\lim_{n\to\infty} \frac{d}{d\theta}\lambda_n(\theta) = \lambda_\alpha(\theta)$ *exists, then*

$$L[n^{\frac{1}{2}}(T_{n,\tau_n} - \lambda_n(\theta)) | P_\theta] \to N(0, \lambda_\alpha^2(\theta)/J_\alpha(\theta)). \tag{3.6}$$

Theorem 3.4. Suppose conditions A1-A6 hold for each $\theta_0 \in \Theta$ *and there exists functions* $\mu_\alpha(\theta)$, $v_\alpha^2(\theta)$ *with* $v_\alpha^2(\theta) > 0, \mu_\alpha'(\theta) \neq 0$ *and continuous on* Θ *such that for each* $\theta \in \Theta$

$$L[n^{\frac{1}{2}}(T_{n,\tau_n} - \mu_\alpha(\theta)) | P_\theta] \to N(0, v_\alpha^2(\theta)). \tag{3.7}$$

Then

$$v_\alpha^2(\theta) \geq (\mu_\alpha'(\theta))^2 / J_\alpha(\theta), \tag{3.8}$$

for almost all $\theta \in \Theta$.

4. *Auxiliary lemmata and proofs of the main theorems.*

The particular choice of local alternatives θ_n of (2.5) originates from the fact that our assumptions ensure $n^{-1}J_{n,\tau_n} \to J_\alpha$. To demonstrate this we first establish two auxiliary results.

Let $\{Y_i;\ i \geq 1\}$ be a sequence if iidrv's with a strictly increasing continuous df H having support R^+. Let $Y_{n,1},\ldots,Y_{n,n}$ be the ordered observations of the sample $Y_1,\ldots,Y_n$ and ν_n a rv with values in $\{1,\ldots,n\}$.

Lemma 4.1. Let $g : R^+ \to R$ *be a measurable function such that* $E|g(Y_1)| < \infty$. *Suppose* $n^{-1}\nu_n \to \alpha \in (0,1]$ *in probability. Then*

$$n^{-1} \sum_{i=1}^{\nu_n} g(Y_{n,i}) \to_{L_1} \int_0^{H^{-1}(\alpha)} g(x)\,dH(x).$$

Proof. Let H_n be the empirical df of $Y_1,\dots,Y_n$. Then

$$n^{-1}\sum_{i=1}^{\nu_n} g(Y_{n,i}) = \int_0^{H_n^{-1}(n^{-1}\nu_n)} g(x)\,dH_n(x) \tag{4.1}$$

where $H_n^{-1}(x) = \inf\{t \in R : H_n(t) \ge x\}$. Therefore

$$n^{-1}\sum_{i=1}^{\nu_n} g(Y_{n,i}) - \int_0^{H^{-1}(\alpha)} g(x)\,dH(x) \tag{4.2}$$

$$= \{\int_0^{H_n^{-1}(n^{-1}\nu_n)} g(x)\,dH_n(x) - \int_0^{H^{-1}(\alpha)} g(x)\,dH_n(x)\} +$$

$$\{\int_0^{H^{-1}(\alpha)} g(x)\,dH_n(x) - \int_0^{H^{-1}(\alpha)} g(x)\,dH(x)\}$$

$$\equiv \gamma_{n,1} + \gamma_{n,2}, \text{ say.}$$

We first consider the case $\alpha < 1$. Now if $I(A)$ is the indicator of A

$$\gamma_{n,2} = \{n^{-1}\sum_{i=1}^{n} g(Y_i)I(Y_i < H^{-1}(\alpha)) - \int_0^{H^{-1}(\alpha)} g(x)\,dH(x)\}$$

and thus by the strong law of large numbers (SLLN) we obtain immediately $\gamma_{n,2} \to_{L_1} 0$. It remains to show $\gamma_{n,1} \to_{L_1} 0$.

Let (Ω,F,P) be the underlying probability space and $\varepsilon > 0$ be arbitrary. Define

$$G_n = \{w \in \Omega : |Y_{n,\tau_n}(w) - H^{-1}(\alpha)| > \varepsilon\}$$

$$E_{n,1} = \{(x,w) \in R^+ \times \Omega : Y_{n,\tau_n} > x \ge H^{-1}(\alpha)\}$$

$$E_{n,2} = \{(x,w) \in R^+ \times \Omega : Y_{n,\tau_n} \le x < H^{-1}(\alpha)\}$$

Then G_n, $E_{n,1}$, $E_{n,2}$ are measurable sets and

$$|\gamma_{n,1}| \le \int_{E_{n,1}\cup E_{n,2}} |g(x)|\,|I(x < Y_{n,\tau_n}) - I(x < H^{-1}(\alpha))|\,dH_n(x)$$

for each $w \in \Omega$. But

$$E(|\gamma_{n,1}|I_{G_n}) \le E(\int_0^\infty |g(x)|I_{G_n}\,dH_n(x)) \le E(I_{G_n} n^{-1}\sum_{i=1}^n |g(Y_i)|).$$

Since $n^{-1}\nu_n \to \alpha$ in probability and H is continuous it follows that $E(I_{G_n}) = P(G_n) \to 0$ and since $n^{-1}\sum_{i=1}^n |g(Y_i)| \to_{L_1} E|g(Y_1)|$ by the SLLN we get

$$E(|\gamma_{n,1}|I_{G_n}) \to 0. \tag{4.3}$$

Now

$$|\gamma_{n,1}|I_{\bar{G}_n} \le \sum_{i=1}^n \int_{E_{n,i}} |g(x)|I_{\bar{G}_n}\,dH_n(x) \equiv \gamma_{n,3} + \gamma_{n,4},$$ say, where $\bar{G}_n$ is the complement of G_n in Ω.

For $(x,w) \in E_{n,1}$, $w \in \bar{G}_n$ we have $H^{-1}(\alpha) \le x < Y_{n,\tau_n} \le H^{-1}(\alpha) + \varepsilon$ and for $(x,w) \in E_{n,2}$, $w \in \bar{G}_n$, $H^{-1}(\alpha) > x \ge Y_{n,\tau_n} > H^{-1}(\alpha) - \varepsilon$. Therefore

$$\gamma_{n,3} \le \int_{H^{-1}(\alpha)}^{H^{-1}(\alpha)+\varepsilon} |g(x)|\,dH_n(x) \to_{L_1} \int_{H^{-1}(\alpha)}^{H^{-1}(\alpha)+\varepsilon} |g(x)|\,dH(x)$$

and similarly $\gamma_{n,4} \le \int_{H^{-1}(\alpha)-\varepsilon}^{H^{-1}(\alpha)} |g(x)|\,dH_n(x) \to_{L_1} \int_{H^{-1}(\alpha)-\varepsilon}^{H^{-1}(\alpha)} |g(x)|\,dH(x).$

Since $E|g(Y)| < \infty$ and $\varepsilon > 0$ is arbitrary we have shown $E(|\gamma_{n,1}|I_{\bar{G}_n}) \to 0$ and so $\gamma_{n,1} \to_{L_1} 0$. This establishes the lemma for the case $\alpha < 1$.

For $\alpha = 1$ we interpret $H^{-1}(\alpha) = +\infty$ and so must show

$$n^{-1}\sum_{i=1}^{\nu_n} g(Y_{n,i}) \to_{L_1} \int_0^\infty g(x)\,dH(x) = Eg(Y).$$

We proceed as in (4.2). The proof of $\gamma_{n,2} \to_{L_1} 0$ is essentially the same and for $\gamma_{n,1}$ we write

$$\gamma_{n,1} = \int_0^\infty g(x)\{I_{E_n} - 1\}dH_n(x)$$

where $E_n = \{(x,w) \in R^+ \times \Omega : Y_{n,\tau_n}(w) > x\}$. Also define

$K_n = \{w \in \Omega : |n^{-1}\tau_n(w) - 1| > \varepsilon\}$. Then as before $\gamma_{n,1} I_{K_n} \to_{L_1} 0$. For

$(x,w) \in E_n$, $w \in \bar{K}_n$ we have $x \geq Y_{n,[n(1-\varepsilon)]}$. So

$$|\gamma_{n,1}| I_{\bar{K}_n} \leq \int_0^\infty |g(x)| I(x \geq Y_{n,[n(1-\varepsilon)]}) dH_n(x) \tag{4.4}$$

$$\leq \int_0^\infty |g(x)| dH_n(x) - n^{-1} \sum_{i=1}^{[n(1-\varepsilon)]} |g(Y_{n,i})| .$$

Therefore from the first part and the SLLN the right hand side of (4.4) converges in L_1 to $\int_0^\infty |g(x)|dH(x) - \int_0^{H^{-1}(1-\varepsilon)} |g(x)|dH(x)$, which can be made arbitrarily small by choosing ε appropriately. This completes the proof.

The next result is based on a theorem by Sen (1961). We assume that H admits a density (with respect to Lebesgue measure μ) h on R^+ and $h(0) > 0$.

<u>Lemma 4.2.</u> *Let* $g : R^+ \to R$ *be right continuous in a neighborhood of the origin and have a right derivative* $g'(0)$ *there. Suppose* $E|g(Y)|^a < \infty$ *for some* $a > 0$. *Then*

$$\lim_{n\to\infty} E\{n^a |g(Y_{n,1}) - g(0)|^a\} = \Gamma(a + 1)\{|g'(0)|/h(0)\}^a .$$

<u>Proof.</u> $E\{n^a|g(Y_{n,1}) - g(0)|^a\} = n^{a+1}\int_0^{x_{n,0}} |g(x) - g(0)|^a (1-H(x))^{n-1} dH(x)$

$$+ n^{a+1}\int_{x_{n,0}}^\infty |g(x) - g(0)|^a (1 - H(x))^{n-1} dH(x)$$

$$\equiv \delta_{n,1} + \delta_{n,2}, \text{ say.}$$

The point $x_{n,0} \in (0,\infty)$ is chosen such that $H(x_{n,0}) = cn^{-\delta}$ where $c > 0$ and $0 < \delta < 1$. Now $(1 - H(x))$ is nonincreasing and so

$$\delta_{n,2} \le n^{a+1}(1 - H(x_{n,0}))^{n-1}\int_{x_{n,0}}^{\infty} |g(x) - g(0)|^a dH(x)$$

$$\le n^{a+1}(1 - H(x_{n,0}))^{n-1}E|g(Y_1) - g(0)|^a$$

But $n^{a+1}(1 - H(x_{n,o}))^n \le n^{a+1}\exp(-cn^{1-\delta}) \to 0$ as $n \to \infty$. So $\delta_{n,2} \to 0$ and we are left with $\delta_{n,1}$. Now $\delta_{n,1} = n^{a+1}\int_0^{cn^{-\delta}} |g(H^{-1}(x)) - g(0)|^a(1 - x)^{n-1}dx$.

We can find $c_n \in (0,cn^{-\delta}]$ such that for sufficiently large n

$$\delta_{n,1} = \{|g(H^{-1}(c_n)) - g(0)|/c_n\}^a n^{a+1}\int_0^{cn^{-\delta}} x^a(1 - x)^{n-1}dx.$$

It is shown in Sen (1961) that $n^{a+1}\int_0^{cn^{-\delta}} x^a(1 - x)^{n-1}dx \to \Gamma(a + 1)$ as $n \to \infty$. Also

$$\lim_{n\to\infty} \{|g(H^{-1}(c_n)) - g(0)|/c_n\} = \lim_{n\to\infty} \{|g(H^{-1}(c_n)) - g(0)|/H^{-1}(c_n)\}$$
$$\cdot \lim_{n\to\infty} \{H^{-1}(c_n)/c_n\} = |g'(0)|/h(0),$$

and thus the conclusion of the lemma follows.

The following lemma is central to the development of our main theorems.

Lemma 4.3. *Suppose assumptions* (A1) - (A3) *hold. Then for any sequence of stopping variables* ν_n *adapted to* $\{\mathcal{B}_{n,k} : 1 \le k \le n\}$ *for which* $n^{-1}\nu_n \to \alpha \in (0,1]$ *in* P_{θ_0}*-probability,*

$$n^{-1}V_{n,\nu_n} \to_{L_1} J_\alpha .$$

Proof. In the sequel we suppress θ_0 throughout. From (2.4) and (2.13) we have $V_{n,\nu_n} = \sum_{i=1}^{\nu_n} \sigma_{n,i}^{*2}$ where for each $1 \le i \le n$,

$$\sigma^{*2}_{n,i} = \dot{r}^2(Z_{i-1}) + \beta_0 + \beta_1 + \beta_2 + \beta_3 \qquad (4.5)$$

and

$$\beta_0 \equiv E\,((\dot{r}^2(Z_i) - \dot{r}^2(Z_{i-1}))\,|\,\mathcal{B}_{n,i-1})$$

$$\beta_1 \equiv E\,((n-i+1)^2(\dot{G}\,(Z_i) - \dot{G}\,(Z_{i-1}))^2\,|\,\mathcal{B}_{n,i-1})$$

$$\beta_2 \equiv 2E\,((n-i+1)(\dot{r}\,(Z_i)\dot{G}\,(Z_i) - \dot{r}\,(Z_{i-1})\dot{G}\,(Z_{i-1}))\,|\,\mathcal{B}_{n,i-1})$$

$$\beta_3 \equiv -2\dot{G}\,(Z_{i-1})E\,((n-i+1)(\dot{r}\,(Z_i) - \dot{r}\,(Z_{i-1}))\,|\,\mathcal{B}_{n,i-1}).$$

Now (A1)and(A2) ensure the continuity of $\dot{G}$ and $\dot{r}$ and the existence of their derivatives at Z_{i-1}. Also $\frac{\partial}{\partial x}\dot{G}\,(x) = -r\,(x)\dot{r}\,(x)$. Furthermore if for each $n \geq i \geq 1$, $Y_1,\ldots,Y_{n-i+1}$ are iidrv's with df given by $\tilde{F}\,(x) = (F\,(x) - F\,(Z_{i-1}))/(1 - F\,(Z_{i-1}))$, if $x > Z_{i-1}$ and zero otherwise, then the conditional distribution of Z_i given $\mathcal{B}_{n,i-1}$ is the same as that of $\min\{Y_1,\ldots,Y_{n-i+1}\}$. Hence applying Lemma 4.2 repeatedly we find that for each $n \geq i \geq 1$ β_0, β_1, β_2, β_3 are respectively convergent equivalent a.s. (P_{θ_0}) to 0, $2\dot{r}^2(Z_{i-1})$, $-2\dot{r}^2(Z_{i-1}) + 2\{r^{-1}(x)\dot{G}\,(x)\,\frac{\partial}{\partial x}\,\dot{r}\,(x)\}_{x=Z_{i-1}}$ and $-2\{\dot{r}^{-1}(x)\dot{G}\,(x)\,\frac{\partial}{\partial x}\,\dot{r}\,(x)\}_{x=Z_{i-1}}$. Hence $\sum_{j=0}^{3}\beta_j$ is convergent equivalent a.s. to 0 and we get $n^{-1}V_{n,\nu_n}$ convergent equivalent a.s. to $n^{-1}\sum_{i=1}^{\nu_n}\dot{r}^2(Z_{i-1})$. An application of Lemma 4.1 yields the desired result.

We shall now turn to the analysis of the statistics $\Lambda_{n,\tau_n}(u)$ with u fixed. In the sequel θ_0 will be held fixed and therefore we suppress θ_0 in P_{θ_0} and E_{θ_0}. This convention will also apply to the ancillary entities to be introduced below. Convergences are to be interpreted with respect to P_{θ_0}.

Define $\{\eta_{n,i} : 1 \leq i \leq n\}$ by

$$\eta_{n,i} \equiv \eta_{n,i}(u) = (g_{n,i}(\theta_n)/g_{n,i}(\theta_0))^{\frac{1}{2}} - 1, \qquad (4.6)$$

where $g_{n,i}(\theta) \equiv g_{n,i}(Z_i,\theta) = q_\theta^{\frac{1}{2}}(Z_i|\mathcal{B}_{n,i-1})$ and θ_n is given by (2.5). Denote differentiation with respect to θ by a prime. If $\|\ \|$ denotes the L_2-norm with respect to the product of Lebesgue measure μ and counting measure on $\{1,\ldots,\tau_n(w)\}$ then

$$\|g'_{n,i}(\theta) - g'_{n,i}(\theta_0)\|^2 = \sum_{i=1}^{\tau_n} \int_{Z_{i-1}}^{\infty} (g'_{n,i}(z,\theta) - g'_{n,i}(z,\theta_0))^2 d\mu(z)$$

$$\leq \sup_{|\theta-\theta_0|\leq|u|n^{-\frac{1}{2}}} \sum_{i=1}^{n} \int_{Z_{i-1}}^{\infty} (g'_{n,i}(z,\theta) - g'_{n,i}(z,\theta_0))^2 d\mu(z),$$

for each u and hence (A6) entails

$$\lim_{n\to\infty} E\{n^{-1} \sup_{|\theta-\theta_0|\leq|u|n^{-\frac{1}{2}}} \|g'_{n,i}(\theta) - g'_{n,i}(\theta_0)\|^2\} = 0. \tag{4.7}$$

We shall utilize the following lemma in the proof of Theorem 3.1.

Lemma 4.4. For each $u \in R$

$$E\left(\sum_{i=1}^{\tau_n} (\eta_{n,i}(u) - \frac{1}{2}\, un^{-\frac{1}{2}}\xi^*_{n,i})^2\right) \to 0 \tag{4.8}$$

and

$$E\left(\sum_{i=1}^{\tau_n} \eta^2_{n,i}(u)\right) \to \frac{1}{4}\, u^2 J_\alpha . \tag{4.9}$$

Proof: Since τ_n is adapted to $\{\mathcal{B}_{n,i} : 1 \leq i \leq n\}$ the expectation in (4.8) may be written

$$E\left(\sum_{i=1}^{\tau_n} E\{(\eta_{n,i} - \frac{1}{2}\, un^{-\frac{1}{2}}\xi^*_{n,i})^2 |\mathcal{B}_{n,i-1}\}\right) \tag{4.10}$$

and the sum of the conditional expectations in (4.10) can be re-expressed as $\|(\eta_{n,i} - \frac{1}{2}\, un^{-\frac{1}{2}}\xi^*_{n,i})g_{n,i}\|^2$. Hence to prove (4.8) we must show

$$E(\|(\eta_{n,i} - \frac{1}{2}\, un^{-\frac{1}{2}}\xi^*_{n,i})g_{n,i}\|^2) \to 0 . \tag{4.11}$$

Observe that for each i, $1 \leq i \leq n$

$$(g_{n,i}(\theta_n) - g_{n,i}(\theta_0)) - un^{-\frac{1}{2}}g'_{n,i} = \int_{\theta_0}^{\theta_n} (g'_{n,i}(\theta) - g'_{n,i}(\theta_0))d\mu(\theta). \quad (4.12)$$

But $g'_{n,i}/g_{n,i} = \frac{1}{2}\xi^*_{n,i}$ and thus (4.12) leads to

$$\|(\eta_{n,i} - \frac{1}{2} un^{-\frac{1}{2}}\xi^*_{n,i})g_{n,i}\|^2 = \sum_{i=1}^{\tau_n} \int_{Z_{i-1}}^{\infty} (\int_{\theta_0}^{\theta_n} (g'_{n,i}(\theta) - g'_{n,i}(\theta_0))d\mu(\theta))d\mu(z). \quad (4.13)$$

An application of the Cauchy-Schwarz inequality and Fubini's theorem in turn on the right hand side of (4.13) will yield after some routine manipulations

$$\|(\eta_{n,i} - \frac{1}{2} un^{-\frac{1}{2}}\xi^*_{n,i})g_{n,i}\|^2 \le u^2 n^{-1} \sup_{|\theta-\theta_0|\le|u|n^{-\frac{1}{2}}} \|g'_{n,i}(\theta) - g'_{n,i}(\theta_0)\|^2 \quad (4.14)$$

and so (4.11) is a consequence of (4.7) and (4.14). Observe that in our notation

$$\|\eta_{n,i}g_{n,i}\|^2 = \sum_{i=1}^{\tau_n} E(\eta^2_{n,i}|\mathcal{B}_{n,i-1}) \quad (4.15)$$

and

$$\|\frac{1}{2} un^{-\frac{1}{2}}\xi^*_{n,i}g_{n,i}\|^2 = \frac{1}{4} u^2 n^{-1} \sum_{i=1}^{\tau_n} E(\xi^{*2}_{n,i}|\mathcal{B}_{n,i-1}) = \frac{1}{4} u^2 n^{-1} V_{n,\tau_n}. \quad (4.16)$$

By Lemma 4.3 $n^{-1}V_{n,\tau_n} \to J_\alpha$ in $L_1(P)$. Therefore from (4.15), (4.16) and the inequality

$$|\|\eta_{n,i}g_{n,i}\| - \|\frac{1}{2} un^{-\frac{1}{2}}\xi^*_{n,i}g_{n,i}\||^2 \le \|(\eta_{n,i} - \frac{1}{2} un^{-\frac{1}{2}}\xi^*_{n,i})g_{n,i}\|^2$$

(4.9) follows from (4.8). The proof of the lemma is now complete.

<u>Proof of Theorem 3.1</u>. From our definition of Λ_{n,τ_n} and $\eta_{n,i}$ we can write by Taylor's theorem

$$\log \Lambda_{n,\tau_n} = 2\sum_{i=1}^{\tau_n} \log(1 + \eta_{n,i}) = 2\sum_{i=1}^{\tau_n} \eta_{n,i} - \sum_{i=1}^{\tau_n} \eta^2_{n,i} + \sum_{i=1}^{\tau_n} \lambda_{n,i}|\eta_{n,i}|^3, \quad (4.17)$$

where $\lambda_{n,i}$ satisfy $|\lambda_{n,i}| < 1$ and $\max_{1\le i\le\tau_n} |\eta_{n,i}| < \varepsilon$, with $\varepsilon > 0$ arbitrary. We rewrite (4.17) in the form

$$\log \Lambda_{n,\tau_n} = \gamma_{n,1} + \gamma_{n,2} + \gamma_{n,3} + \gamma_{n,4} \tag{4.18}$$

where

$$\gamma_{n,1} = 2\{\sum_{i=1}^{\tau_n} \eta_{n,i} - \frac{1}{2}\, un^{-\frac{1}{2}} \sum_{i=1}^{\tau_n} \xi^*_{n,i} + \frac{1}{8}\, u^2 J_\alpha\}, \tag{4.19}$$

$$\gamma_{n,2} = -\{\sum_{i=1}^{\tau_n} \eta^2_{n,i} - \frac{1}{4}\, u^2 J_\alpha\}, \tag{4.20}$$

$$\gamma_{n,3} = \sum_{i=1}^{\tau_n} \lambda_{n,i} |\eta_{n,i}|^3, \tag{4.21}$$

$$\gamma_{n,4} = \{un^{-\frac{1}{2}} J_{n,\tau_n} W_{n,\tau_n} - \frac{1}{2}\, u^2 J_\alpha\}. \tag{4.22}$$

Hence from Lemma 4.3, (3.2) will be established once we show $\gamma_{n,i} \xrightarrow{P} 0$ for $i = 1,2,3$ and $L(W_{n,\tau_n} | P_{\theta_0}) \to N(0,1)$. We begin with the proof of the latter.

In view of (A3) and (A5) we have $E(\xi^*_{n,i} | \mathcal{B}_{n,i-1}) = 0$ for each i and so with $\xi_{n,k}$ given by (2.9), $\{\xi_{n,k}, \mathcal{B}_{n,k} : 1 \le k \le n\}$ is a zero-mean martingale under P. From Lemma 4.3 and (A4)

$$J^{-1}_{n,\tau_n} \sum_{i=1}^{\tau_n} E(\xi^{*2}_{n,i} | \mathcal{B}_{n,i-1}) = (n^{-1} J_{n,\tau_n})^{-1} (n^{-1} V_{n,\tau_n}) \xrightarrow{P} 1 \tag{4.23}$$

Hence by Durrett and Resnick (1978) Theorem 2.3, we will have $W_{n,\tau_n} \xrightarrow{L} N(0,1)$ once we establish

$$J^{-1}_{n,\tau_n} \sum_{i=1}^{\tau_n} E(\xi^{*2}_{n,i}\, I(|\xi^*_{n,i}| > \varepsilon\, J^{\frac{1}{2}}_{n,\tau_n}) | \mathcal{B}_{n,i-1}) \xrightarrow{P} 0 \tag{4.24}$$

for arbitrary $\varepsilon > 0$. Now the entity in (4.24) is dominated by

$$(n^{-1} J_{n,\tau_n})^{-(1+\delta/2)} \{n^{-1} \sum_{i=1}^{\tau_n} E(|\xi^*_{n,i}|^{2+\delta} | \mathcal{B}_{n,i-1})\} (n\varepsilon)^{-\delta}, \tag{4.25}$$

where the δ comes from (A3). Therefore in view of (A4) and Lemma 4.3, (4.24) will be established once we show

$$\overline{\lim_{n\to\infty}}\, n^{-1} E\left(\sum_{i=1}^{\tau_n} |\xi^*_{n,i}|^{2+\delta}\right) < \infty \tag{4.26}$$

With our definition of $\xi^*_{n,i}$ in (2.12) we have

$$|\xi^*_{n,i}|^{2+\delta} \leq 2^{1+\delta}\{|\dot{r}(Z_i)|^{2+\delta} + |(n-i+1)(\dot{G}(Z_i) - \dot{G}(Z_{i-1}))|^{2+\delta}\} \tag{4.27}$$

Now summoning Lemma 4.2 and following the argument in Lemma 4.3 we note that for $n \geq i \geq 1$

$$E\{|(n-i+1)(\dot{G}(Z_i) - \dot{G}(Z_{i-1}))|^{2+\delta}|\mathcal{B}_{n,i-1}\}$$

is convergent equivalent a.s. to $\Gamma(3+\delta)|\dot{r}(Z_{i-1})|^{2+\delta}$. Therefore for arbitrary $\varepsilon > 0$

$$\begin{aligned} & n^{-1}E\left(\sum_{i=1}^{\tau_n} |(n-i+1)(\dot{G}(Z_i) - \dot{G}(Z_{i-1}))|^{2+\delta}\right) \\ & \leq n^{-1}E\left(\sum_{i=1}^{\tau_n} \{\Gamma(3+\delta)|\dot{r}(Z_{i-1})|^{2+\delta}\} + \varepsilon n\right) \\ & \leq \Gamma(3+\delta)E|\dot{r}(X)|^{2+\delta} + \varepsilon, \end{aligned} \tag{4.28}$$

and thus from (4.27), (4.28) and (A3), (4.26) entails.

We now consider $\gamma_{n,2}$ and $\gamma_{n,3}$. That $\gamma_{n,i} \xrightarrow{P} 0$ for $i = 2,3$ will follow from

$$\max_{1\leq i\leq \tau_n} |\eta_{n,i}| \xrightarrow{P} 0 \tag{4.29}$$

and

$$\sum_{i=1}^{\tau_n} \eta^2_{n,i} \xrightarrow{P} \frac{1}{4}\, u^2 J_\alpha \tag{4.30}$$

To show this let $\varepsilon > 0$ and $\eta > 0$ be arbitrary. From Durrett and Resnick (1978) we have the inequality

$$P(\max_{1\le i\le \tau_n} |\eta_{n,i}| > \varepsilon) < \eta + P(\sum_{i=1}^{\tau_n} P(|\eta_{n,i}| > \varepsilon|\mathcal{B}_{n,i-1}) > \eta). \tag{4.31}$$

But
$$\begin{aligned} P(|\eta_{n,i}| > \varepsilon|\mathcal{B}_{n,i-1}) &\le P(|\eta_{n,i} - \tfrac{1}{2}\, un^{-\frac12}\xi^*_{n,i}| > \varepsilon/2|\mathcal{B}_{n,i-1}) \\ &\quad + P(|\xi^*_{n,i}| > |u|^{-1}\varepsilon n^{-\frac12}|\mathcal{B}_{n,i-1}) \\ &\le 4\varepsilon^{-2}E((\eta_{n,i} - \tfrac{1}{2}\, un^{-\frac12}\xi^*_{n,i})^2|\mathcal{B}_{n,i-1}) \\ &\quad + u^2\varepsilon^{-2}n^{-1}E(\xi^{*2}_{n,i}I(|\xi^*_{n,i}| > |u|^{-1}\varepsilon n^{-\frac12})|\mathcal{B}_{n,i-1}) \end{aligned} \tag{4.32}$$

Now (4.24) holds. So from Lemma 4.3 we get

$$n^{-1}\sum_{i=1}^{\tau_n} E(\xi^{*2}_{n,i}I(|\xi^*_{n,i}| > |u|^{-1}\varepsilon n^{-\frac12})|\mathcal{B}_{n,i-1}) \xrightarrow{p} 0. \tag{4.33}$$

Furthermore from (4.8) $\sum_{i=1}^{\tau_n} E((\eta_{n,i} - \frac{1}{2}\, un^{-\frac12}\xi^*_{n,i})^2|\mathcal{B}_{n,i-1}) \xrightarrow{p} 0$ and therefore (4.29) obtains. Again with $\varepsilon,\eta > 0$ arbitrary,

$$\begin{aligned} P(|\sum_{i=1}^{\tau_n} \eta^2_{n,i} - \tfrac{1}{4}\, u^2n^{-1}\sum_{i=1}^{\tau_n} \xi^{*2}_{n,i}| > \varepsilon) &\le \varepsilon^{-1}E(\sum_{i=1}^{\tau_n} |\eta^2_{n,i} - \tfrac{1}{2}\, u^2n^{-1}\xi^{*2}_{n,i}|) \\ \le \tfrac{1}{2}\,\varepsilon^{-1}\eta\, E(\sum_{i=1}^{\tau_n} (\eta_{n,i} - \tfrac{1}{2}\, un^{-\frac12}\xi^*_{n,i})^2) &+ \varepsilon^{-1}\eta^{-1}(E(\sum_{i=1}^{\tau_n} \eta^2_{n,i} + \tfrac{1}{4}\, u^2n^{-1}J_{n,\tau_n})). \end{aligned} \tag{4.34}$$

Select $\eta = \{E(\sum_{i=1}^{\tau_n} (\eta_{n,i} - \frac{1}{2}\, un^{-\frac12}\xi^*_{n,i})^2)\}^{-\frac12}$ and apply Lemma 4.3 and 4.4. We get from (4.34)

$$(\sum_{i=1}^{\tau_n} \eta^2_{n,i} - \tfrac{1}{4}\, u^2n^{-1}\sum_{i=1}^{\tau_n} \xi^{*2}_{n,i}) \xrightarrow{p} 0\,, \tag{4.35}$$

and then using McLeish (1974) (Theorem 3.6 and Corollary 3.8) we get $n^{-1}\sum_{i=1}^{\tau_n} \xi^{*2}_{n,i} \xrightarrow{p} J_\alpha$ and then (4.30) follows from (4.35).

It remains to show $\gamma_{n,1} \xrightarrow{p} 0$. Note that (4.34) and (4.35) also imply

$$\sum_{i=1}^{\tau_n} E(\eta^2_{n,i}|\mathcal{B}_{n,i-1}) \xrightarrow{p} \tfrac{1}{4}\, u^2J_\alpha \tag{4.36}$$

and therefore directly from (4.6)

$$\sum_{i=1}^{\tau_n} E(\eta_{n,i}|\mathcal{B}_{n,i-1}) \xrightarrow{P} -\frac{1}{8} u^2 J_\alpha . \tag{4.37}$$

For arbitrary $\varepsilon > 0$

$$P(|\sum_{i=1}^{\tau_n} \eta_{n,i} - \frac{1}{2} un^{-\frac{1}{2}} \sum_{i=1}^{\tau_n} \xi^*_{n,i} + \frac{1}{8} u^2 J_\alpha| > \varepsilon)$$

$$\leq P(|\sum_{i=1}^{n} \zeta_{n,i}| > \varepsilon/2) + P(|\sum_{i=1}^{\tau_n} E(\eta_{n,i}|\mathcal{B}_{n,i-1}) + \frac{1}{8} u^2 J_\alpha|) \tag{4.38}$$

where $\zeta_{n,i} = \eta_{n,i} - \frac{1}{2} un^{-\frac{1}{2}}\xi^*_{n,i} - E(\eta_{n,i}|\mathcal{B}_{n,i-1})$, $1 \leq i \leq n$.

Now $\{\zeta_{n,i}, \mathcal{B}_{n,i}; 1 \leq i \leq n\}$ is a zero-mean martingale. So we have

$$P(|\sum_{i=1}^{\tau_n} \zeta_{n,i}| > \varepsilon/2) \leq 4\varepsilon^{-2} E(\sum_{i=1}^{\tau_n} \zeta_{n,i})^2 \tag{4.39}$$

and

$$\begin{aligned} E(\sum_{i=1}^{\tau_n} \zeta_{n,i})^2 &\leq E(\sum_{i=1}^{\tau_n} \zeta^2_{n,i}) \\ &= E(\sum_{i=1}^{\tau_n} E(\zeta^2_{n,i}|\mathcal{B}_{n,i-1})) \\ &\leq E(\sum_{i=1}^{\tau_n} (\eta_{n,i} - \frac{1}{2} un^{-\frac{1}{2}}\xi^*_{n,i})^2) \end{aligned} \tag{4.40}$$

That $\gamma_{n,1} \xrightarrow{P} 0$ now follows from (4.38), (4.39), (4.40) and (4.8). The proof of Theorem 3.1 is now complete.

<u>Proof of Theorem 3.2</u>. Observe that (3.3) is an immediate consequence of Theorem 3.1. We also note that this entails

$$L(\log \Lambda_{n,\tau_n}(u)|P_{n,\theta_0}) \to N(-\tfrac{1}{2}\sigma^2, \sigma^2) \tag{4.41}$$

where $\sigma^2 = u^2 J_\alpha$ and P_{n,θ_0} as before is the restriction of P_{θ_0} to $\mathcal{B}_{n,\tau_n}$. Hence by LeCam's First Lemma (see Hájek and Sidak (1967)) it follows that the family of probability measures $\{P_{n,\theta_n} : n \geq 1\}$ is contiguous to $\{P_{n,\theta_0} : n \geq 1\}$ and so we have for each $u \in R$

$$L(\log \Lambda_{n,\tau_n}(u)|P_{n,\theta_n}) \to N(\tfrac{1}{2}\sigma^2,\sigma^2) \tag{4.42}$$

from which (3.4) follows.

Proof of Theorem 3.3. Since

$$\lambda_n(\theta) = \sum_{k=1}^{n}\int_{E_{n,k}} T_{n,k}(\underset{\sim}{z}^{(k)})p_\theta(\underset{\sim}{z}^{(k)},n)d\mu_k(\underset{\sim}{z}^{(k)}),$$

we obtain

$$\frac{d}{d\theta}\lambda_n(\theta) = \sum_{k=1}^{n}\int_{E_{n,k}} T_{n,k}(\underset{\sim}{z}^{(k)})\xi_{n,k}(\underset{\sim}{z}^{(k)},\theta)p_\theta(\underset{\sim}{z}^{(k)},n)d\mu_k(\underset{\sim}{z}^{(k)}) \tag{4.43}$$

$$= E_\theta(T_{n,\tau_n}\xi_{n,\tau_n}).$$

We have already noted that our assumptions imply $E_\theta(\xi_{n,\tau_n}) = 0$. Thus (4.43) can be rewritten

$$E_\theta(\xi_{n,\tau_n}(T_{n,\tau_n} - \lambda_n(\theta))) = \frac{d}{d\theta}\lambda_n(\theta) \equiv \lambda_n'(\theta) \tag{4.44}$$

and an application of the Cauchy-Schwarz inequality yields (3.5). When equality obtains in (3.5) there is a constant $a_n(\theta)$ such that

$$T_{n,\tau_n} - \lambda_n(\theta) = a_n(\theta)\xi_{n,\tau_n}. \tag{4.45}$$

Hence $a_n^2(\theta) = (\lambda_n'(\theta))^2/J_{n,\tau_n}(\theta))^2$. From Lemma 4.3 and the fact that $L(W_{n,\tau_n}|P_{\theta_0}) \to N(0,1)$ the conclusion of the theorem follows from (4.45).

Proof of Theorem 3.4. Suppose the sequence $\{T_{n,\tau_n}\}$ satisfies the condition

$$L[n^{\frac{1}{2}}(T_{n,\tau_n} - \theta_0)|P_{\theta_0}] \to N(0,v_\alpha^2(\theta_0)), \tag{4.46}$$

where $v_\alpha^2(\theta_0) > 0$, and in addition the restriction

$$\liminf_{n\to\infty} P_{\theta_0+n^{-\frac{1}{2}}}[T_{n,\tau_n} < \theta_0 + n^{-\frac{1}{2}}] \le \tfrac{1}{2}. \tag{4.47}$$

We shall show $v_\theta^2(\theta_0) \ge J_\alpha^{-1}(\theta_0)$.

Let $\varepsilon > 0$ be arbitrary. Set $\theta_n = \theta_0 + n^{-\frac{1}{2}}$ and define the sets $C_n = [\lambda_{n,\tau_n} > \varepsilon]$, $D_n = [T_{n,\tau_n} \ge \theta_n]$, $n \ge 1$ where $\lambda_{n,\tau_n} = \log \Lambda_{n,\tau_n}(1)$. Then in view of (4.47),

$$\limsup_{n\to\infty} P_{\theta_n}(D_n) \ge \tfrac{1}{2}. \tag{4.48}$$

Also

$$P_{\theta_n}(C_n) = 1 - P_{\theta_n}[(\lambda_{n,\tau_n} - \tfrac{1}{2}\sigma^2)/\sigma \le (\varepsilon - \tfrac{1}{2}\sigma^2)/\sigma]$$

where $\sigma^2 = J_\alpha(\theta_0)$. Hence from Theorem 3.2 we get

$$\lim_{n\to\infty} P_{\theta_n}(C_n) = 1 - \Phi((\varepsilon - \tfrac{1}{2}\sigma^2)/\sigma) \tag{4.49}$$

where Φ is the standard Gaussian distribution function. Hence selecting $\varepsilon > \frac{1}{2}\sigma^2$, (4.48) and (4.49) lead to the inequalities

$$\limsup P_{\theta_n}(D_n) \ge \tfrac{1}{2} > \lim P_{\theta_n}(C_n). \tag{4.50}$$

Thus for infinitely many n,

$$P_{\theta_n}(D_n) > P_{\theta_n}(C_n). \tag{4.51}$$

By the Neyman-Pearson Lemma the test based on λ_{n,τ_n} is the most powerful test of its size among all tests whose stopping variable is τ_n. So we have from (4.51)

$$P_{\theta_0}(D_n) > P_{\theta_0}(C_n) \tag{4.52}$$

for infinitely many n. But

$$P_{\theta_0}(D_n) = 1 - P_{\theta_0}[n^{\frac{1}{2}}(T_{n,\tau_n} - \theta_0)/v_\alpha(\theta_0) < v_\alpha^{-1}(\theta_0)]$$

and

$$P_{\theta_0}(C_n) = 1 - P_{\theta_0}[(\lambda_{n,\tau_n} + \tfrac{1}{2}\sigma^2)/\sigma \le (\varepsilon + \tfrac{1}{2}\sigma^2)/\sigma].$$

Hence (4.51) implies $\Phi((\varepsilon + \frac{1}{2}\sigma^2)/\sigma) \ge \Phi(v_\alpha^{-1}(\theta_0))$, in view of (4.46) and Theorem 3.2. Since $\varepsilon > \frac{1}{2}\sigma^2$ but is otherwise arbitrary we get the result $v_\alpha^2(\theta_0) \ge \sigma^{-2} = J_\alpha^{-1}(\theta_0)$.

Now suppose (4.46) holds with θ_0 replaced by θ, for any $\theta \in \Theta$. The function given by $g_n(\theta) = |P_\theta[T_{n,\tau_n} < \theta] - \frac{1}{2}|$, $\theta \in \Theta$ and zero otherwise, is Borel-measurable (with respect to the Borel subsets of Θ) and our assumption implies $\lim_{n\to\infty} P_\theta(T_{n,\tau_n} < \theta) = \frac{1}{2}$, for all $\theta \in \Theta$. Hence

$$\lim_{n\to\infty} g_n(\theta) = 0 \quad \text{and} \quad 0 \le g_n(\theta) \le \tfrac{1}{2}, \text{ for each } \theta \in \Theta.$$

Therefore since

$$\int_R g_n(\theta + n^{-\frac{1}{2}})d\Phi(\theta) = \int_R g_n(\theta)\frac{1}{\sqrt{2\pi}}\exp(-\tfrac{1}{2}(\theta - n^{-\frac{1}{2}})^2)d\mu(\theta),$$

the dominated convergence theorem yields

$$\lim_{n\to\infty}\int_R g_n(\theta + n^{-\frac{1}{2}})d\Phi(\theta) = 0.$$

It follows that $g_{n_\nu}(\theta + n_\nu^{-\frac{1}{2}}) \to 0$ a.e. (Φ) along some subsequence (n_ν). But the measure in $(R,\mathcal{B})$ induced by Φ is equivalent to Lebesgue measure μ. So for almost all $\theta \in \Theta$ (respect to μ) we have $\liminf_{n\to\infty} g_n(\theta + n^{-\frac{1}{2}}) = 0$ and thus also $\liminf_{n\to\infty} P_{\theta+n^{-\frac{1}{2}}}[T_{n,\tau_n} < \theta + n^{-\frac{1}{2}}] \le \frac{1}{2}$, for almost all $\theta \in \Theta$, and so the conclusion $v_\alpha^2(\theta) \ge J_\alpha^{-1}(\theta)$ can be made for almost all $\theta \in \Theta$. The stated form of Theorem 3.4 will now follow with only minor modifications.

5. Concluding Remarks.

The restriction to nonnegative random variables made at the beginning of this paper is unnecessary and our results will continue to hold with minor modifications in more general cases. With the appropriate changes for instance our results will hold true for distributions having a finite right end-point.

The particular choice of local coordinates $\theta_n = \theta + un^{-\frac{1}{2}}$ of (2.5) in the definition of the PCLRS $\{\Lambda_{n,k}\}$ in (2.6) was a consequence of the convergence $n^{-1}J_{n,\tau_n} \to J_\alpha$ deduced from Lemma 4.3 and (A4). It was at this stage where the consideration of the observables $\{Z_i\}$ as order statistics came into play.

The proof of the main result Theorem 3.1 reveals two basic features. Firstly we derive the limiting normal distribution for the sequence of derivatives of the log-likelihood $\{\xi_{n,k}(\theta_0) : 1 \leq k \leq n\}$ and then analyze terms of a particular Taylor expansion of the log-likelihood ratios $\{p_{\theta_n}(\underset{\sim}{Z}^{(k)},n)/p_{\theta_0}(\underset{\sim}{Z}^{(k)},n) : 1 \leq k \leq n\}$. In this expansion the weak continuity condition (A6) makes third order terms negligible in probability. The basic tools utilized in the derivation of the limiting distribution of W_{n,τ_n} of (2.14) were the martingale character of $\{\xi_{n,k}(\theta_0), \mathcal{B}_{n,k} : 1 \leq k \leq n\}$ which is a consequence of (A5), the existence of the limit for stopped suitably normalized conditional variances V_{n,τ_n}, as proved in Lemma 4.3 and the (conditional) Lindeberg condition (4.24) which we derived using (A3).

With these remarks in mind we outline below a set of conditions under which the local asymptotic normality can be obtained for likelihood ratio statistics where the underlying observations follow a series scheme.

Suppose $\{X_{n,k} : 1 \leq k \leq n;\ n \geq 1\}$ is a double sequence of random variables on a probability space (X,A,P_θ) where $\theta \in \Theta$ and Θ is an open

subset of R. Let $\underset{\sim}{X}_{n,k} = (X_{n,1},\ldots,X_{n,k})$, $1 \le k \le n$ and $\mathcal{B}_{n,k}$ denote the σ-field generated by $\underset{\sim}{X}_{n,k}$. The projection of P_θ to $\mathcal{B}_{n,k}$ is denoted $P^{(k)}_{n,\theta}$ and we suppose there is some σ-finite measure μ on $(\mathcal{X},\mathcal{A})$ such that $P^{(k)}_{n,\theta}$ is absolutely continuous with respect to the product $\mu_k = \mu \times \mu \times \ldots \times \mu$ on the cartesian product $(\mathcal{X}^k,\mathcal{A}^k)$. As before, for each $n \ge 1$ let τ_n be a stopping variable adapted to $\{\mathcal{B}_{n,k} : 1 \le k \le n\}$ and let $P_{n,\theta}$ denote the restriction of P_θ to $\mathcal{B}_{n,\tau_n} = \sigma(\underset{\sim}{X}_{n,\tau_n})$.

If $p_\theta(\underset{\sim}{x}_{n,k};n)$ is the pdf of $\underset{\sim}{X}_{n,k}$ and $q_\theta(x_{n,k}|\mathcal{B}_{n,k-1})$ the conditional pdf of $X_{n,k}$ given $\mathcal{B}_{n,k-1}$ we have

$$p_\theta(\underset{\sim}{X}_{n,k}\,;\,n) = \prod_{i=1}^{k} q_\theta(X_{n,i}|\mathcal{B}_{n,i-1}),\ 1 \le k \le n. \tag{5.1}$$

Let θ_0 be a fixed point in Θ. We assume

(B1) For all θ in some neighborhood N_{θ_0} of θ_0 and all $\underset{\sim}{x}_{n,k}$, $p_\theta(\underset{\sim}{x}_{n,k};n) > 0$, $1 \le k \le n$; $\theta \to p_\theta(\underset{\sim}{x}_{n,k};n)$ is continuously differentiable on N_{θ_0} for μ_k-almost all $\underset{\sim}{x}_{n,k}$.

(B2) For each $n \ge 1$ and all k, $\theta \to \int_{\mathcal{X}} q_\theta(x|\mathcal{B}_{n,k-1})d\mu$ is differentiable under the integral sign at θ_0.

We may now define for $1 \le k \le n$,

$$\xi_{n,k} = \left[\frac{\partial}{\partial\theta} \log p_\theta(\underset{\sim}{X}_{n,k};n)\right]_{\theta=\theta_0} \tag{5.2}$$

$$\xi^*_{n,k} = \left[\frac{\partial}{\partial\theta} \log q_\theta(X_{n,k}|\mathcal{B}_{n,k-1})\right]_{\theta=\theta_0}. \tag{5.3}$$

We also assume

(B3) For each $n \ge 1$ and all k, $0 < E_{\theta_0}(\xi^{*2}_{n,k}) < \infty$. Let us then define

$$V_{n,k} = \sum_{i=1}^{k} E_{\theta_0}(\xi^{*2}_{n,i}|\mathcal{B}_{n,i-1}),\ 1 \le k \le n \tag{5.4}$$

and suppose

(B4) There exists a sequence of positive constants $\{\Psi_n : n \geq 1\}$ such that $V_{n,\tau_n}/\Psi_n \to 1$ in P_{θ_0}-probability.

(B5) For all $\varepsilon > 0$

$$\Psi_n^{-1} \sum_{i=1}^{\tau_n} E_{\theta_0}(\xi_{n,i}^{*2} I(|\xi_{n,i}^{*}| > \varepsilon \Psi_n^{\frac{1}{2}}) | \mathcal{B}_{n,i-1}) \to 0$$

in P_{θ_0}-probability.

(B6) For each $u \in R$

$$\lim_{n\to\infty} E_{\theta_0} \{ \sup_{|\theta-\theta_0| \leq |u| \Psi_n^{-\frac{1}{2}}} \Psi_n^{-1} \sum_{i=1}^{\tau_n} \int_X d\mu [\frac{\partial}{\partial\theta}\{q_\theta(x|\mathcal{B}_{n,i-1})\}^{\frac{1}{2}} - \frac{\partial}{\partial\theta_0}\{q_{\theta_0}(x|\mathcal{B}_{n,i-1})\}^{\frac{1}{2}}]^2\} = 0$$

where $\frac{\partial}{\partial\theta_0}\{q_{\theta_0}(x|\mathcal{B}_{n,i-1})\}^{\frac{1}{2}} = (\frac{\partial}{\partial\theta}\{q_\theta(x|\mathcal{B}_{n,i-1})\}^{\frac{1}{2}})_{\theta=\theta_0}$.

In view of our assumptions we now define

$$\Lambda_{n,k}(u) = p_{\theta_n}(\underset{\sim}{X}_{n,k};n)/p_{\theta_0}(\underset{\sim}{X}_{n,k};n), \; 1 \leq k \leq n \tag{5.5}$$

where $\theta_n = \theta_0 + u\Psi_n^{-\frac{1}{2}} \in \Theta$, $u \in R$ and set

$$\Lambda_{n,\tau_n}(u) = \Lambda_{n,k}(u), \text{ if } \tau_n = k \tag{5.6}$$

Hence paralleling Theorem 3.1 we state

Theorem 5.1. With the definitions (5.1)-(5.6) and under conditions B1-B6, for each $u \in R$

$$\Lambda_{n,\tau_n}(u) = \exp\{u\Delta_n - \frac{1}{2} u^2 + \delta_n\}$$

where $\Delta_n = \xi_{n,\tau_n}/\Psi_n^{\frac{1}{2}}$ and $L(\Delta_n|P_{\theta_0}) \to N(0,1)$ and $\delta_n(u) \to 0$ in P_{θ_0}-probability for each $u \in R$. We may therefore say the family of probability measures $\{P_{n,\theta} : \theta \in \Theta\}$ is LAN at θ_0.

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Let $X_{n,1} \le X_{n,2} \le \cdots \le X_{n,n}$ be the ordered variables corresponding to a random sample of size n with respect to a family of probability measures $\{P_\theta: \theta \in \Theta\}$ where Θ is an open subset of the real line. In many practical situations the $X_{n,i}$ are the observables and experimentation must be curtailed prior to $X_{n,n}$. If τ_n is a stopping variable adapted

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to the σ-fields $\{\sigma(X_{n,1},\ldots,X_{n,k}) : 1 \leq k \leq n\}$ and $P_{n,\theta}$ the projection of P_θ onto $\sigma(X_{n,1},\ldots,X_{n,\tau_n})$, the local asymptotic normality of the stopped progressively censored likelihood ratio statistics $\Lambda_{n,\tau_n} = dP_{n,\theta_n}/dP_{n,\theta}$ is established with θ, $\theta_n = \theta + un^{-1/2} \in \Theta$ and θ,u held fixed, under certain conditions on the underlying distribution and on τ_n. Conditions are also given to ensure the local asymptotic normality of likelihood ratio statistics where the underlying observations are given in a series scheme.